

A REMARK ON THE PAPER “A UNIFIED PIETSCH DOMINATION THEOREM”

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ABSTRACT. In this short communication we show that the Unified Pietsch Domination proved in [1] remains true even if we remove two of its apparently crucial hypothesis.

1. INTRODUCTION

Let X, Y and E be (arbitrary) non-void sets, \mathcal{H} be a family of mappings from X to Y , G be a Banach space and K be a compact Hausdorff topological space. Let

$$R: K \times E \times G \longrightarrow [0, \infty) \text{ and } S: \mathcal{H} \times E \times G \longrightarrow [0, \infty)$$

be arbitrary mappings.

A mapping $f \in \mathcal{H}$ is said to be RS -abstract p -summing if there is a constant $C > 0$ so that

$$(1.1) \quad \left(\sum_{j=1}^m S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left(\sum_{j=1}^m R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}},$$

for all $x_1, \dots, x_m \in E$, $b_1, \dots, b_m \in G$ and $m \in \mathbb{N}$.

The main result of [1] proves that under certain assumptions on R and S there is a quite general Pietsch Domination-type Theorem. More precisely R and S must satisfy the three properties below:

(1) For each $f \in \mathcal{H}$, there is a $x_0 \in E$ such that

$$R(\varphi, x_0, b) = S(f, x_0, b) = 0$$

for every $\varphi \in K$ and $b \in G$.

(2) The mapping

$$R_{x,b}: K \longrightarrow [0, \infty) \text{ defined by } R_{x,b}(\varphi) = R(\varphi, x, b)$$

is continuous for every $x \in E$ and $b \in G$.

(3) For every $\varphi \in K, x \in E, 0 \leq \eta \leq 1, b \in G$ and $f \in \mathcal{H}$, the following inequalities hold:

$$R(\varphi, x, \eta b) \leq \eta R(\varphi, x, b) \text{ and } S(f, x, b) \leq S(f, x, \eta b).$$

The Pietsch Domination Theorem from [1] reads as follows:

Theorem 1.1. *If R and S satisfy (1), (2) and (3) and $0 < p < \infty$, then $f \in \mathcal{H}$ is RS -abstract p -summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on K such that*

$$(1.2) \quad S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^p d\mu \right)^{\frac{1}{p}}$$

for all $x \in E$ and $b \in G$.

The aim of this note is to show that, surprisingly, the hypothesis (1) and (3) are not necessary. So, Theorem 1.1 is true for arbitrary S (no hypothesis is needed) and the map R just needs to satisfy (2).

D. Pellegrino by INCT-Matematica, PROCAD-NF Capes, CNPq Grant 620108/2008-8 (Ed. Casadinho) and CNPq Grant 301237/2009-3.

2. A RECENT APPROACH TO PDT

In a recent preprint [3] we have extended the Pietsch Domination Theorem from [1] to a more abstract setting, which allows to deal with more general nonlinear mappings in the cartesian product of Banach spaces. In the present note we shall recall the argument used in [3] and a combination of this argument with an interesting argument due to M. Mendel and G. Schechtman (used in [1]) will help us to show that Theorem 1.1 is valid without the hypothesis **(1)** and **(3)** on R and S .

The first step is to prove Theorem 1.1 without the hypothesis **(1)**. This result is proved in [3] in a more general setting. Since the paper [3] is unpublished and we just need a very particular case, we prefer to sketch the proof for this particular case. The proof of this particular case is essentially Pietsch's original proof on a nonlinear disguise.

Theorem 2.1. *Suppose that R and S satisfy **(2)** and **(3)**. A map $f \in \mathcal{H}$ is RS -abstract p -summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on K such that*

$$(2.1) \quad S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^p d\mu \right)^{1/p}$$

for all $x \in E$ and $b \in G$.

Proof. If (2.1) holds it is easy to show that f is RS -abstract p -summing. For the converse, consider the (compact) set $P(K)$ of the probability measures in $C(K)^*$ (endowed with the weak-star topology). For each $(x_j)_{j=1}^m$ in E , $(b_j)_{j=1}^m$ in G and $m \in \mathbb{N}$, let $g : P(K) \rightarrow \mathbb{R}$ be defined by

$$g(\mu) = \sum_{j=1}^m \left[S(f, x_j, b_j)^p - C^p \int_K R(\varphi, x_j, b_j)^p d\mu \right]$$

and \mathcal{F} be the set of all such g . Using **(3)**, one can prove that the family \mathcal{F} is concave and each $g \in \mathcal{F}$ is convex and continuous. Besides, for each $g \in \mathcal{F}$ there is a measure $\mu_g \in P(K)$ such that $g(\mu_g) \leq 0$. In fact, from **(2)** there is a $\varphi_0 \in K$ so that

$$\sum_{j=1}^m R(\varphi_0, x_j, b_j)^p = \sup_{\varphi \in K} \sum_{j=1}^m R(\varphi, x_j, b_j)^p$$

and, considering the Dirac measure $\mu_g = \delta_{\varphi_0}$, we have $g(\mu_g) \leq 0$. So, Ky Fan Lemma asserts that there exists a $\bar{\mu} \in P(K)$ so that

$$g(\bar{\mu}) \leq 0$$

for all $g \in \mathcal{F}$ and by choosing an arbitrary g with $m = 1$ the proof is done. \square

3. THE MAIN RESULT

Note that if each λ_j is a positive integer, by considering each x_j repeated λ_j times in (1.1) one can easily see that (1.1) is equivalent to

$$(3.1) \quad \left(\sum_{j=1}^m \lambda_j S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left(\sum_{j=1}^m \lambda_j R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}}$$

for all $x_1, \dots, x_m \in E$, $b_1, \dots, b_m \in G$, positive integers λ_j and $m \in \mathbb{N}$. Then it is possible to show that (1.1) holds for positive rationals and finally extend to positive real numbers λ_j (using an argument of density). The essence of this argument appears in [1, 2] and is credited to M. Mendel and G. Schechtman.

Now using (3.1) and invoking Theorem 2.1 we can prove a Pietsch Domination-type theorem with no hypothesis on S and just supposing that R satisfies **(2)**:

Theorem 3.1. *Suppose that S is arbitrary and R satisfies (2). A map $f \in \mathcal{H}$ is RS -abstract p -summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on K such that*

$$(3.2) \quad S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^p d\mu \right)^{1/p}$$

for all $x \in E$ and $b \in G$.

Proof. It is clear that if f satisfies (3.2) then $f \in \mathcal{H}$ is RS -abstract p -summing. Conversely, if $f \in \mathcal{H}$ is RS -abstract p -summing, then

$$(3.3) \quad \left(\sum_{j=1}^m \lambda_j S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left(\sum_{j=1}^m \lambda_j R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}}$$

for all $x_1, \dots, x_m \in E$, $b_1, \dots, b_m \in G$, $\lambda_1, \dots, \lambda_m \in [0, \infty)$ and $m \in \mathbb{N}$. Let

$$E_1 = E \times G \quad \text{and} \quad G_1 = \mathbb{K}$$

and define

$$\overline{R}: K \times E_1 \times G_1 \longrightarrow [0, \infty) \quad \text{and} \quad \overline{S}: \mathcal{H} \times E_1 \times G_1 \longrightarrow [0, \infty)$$

by

$$\overline{R}(\varphi, (x, b), \lambda) = |\lambda| R(\varphi, x, b) \quad \text{and} \quad \overline{S}(f, (x, b), \lambda) = |\lambda| S(f, x, b).$$

From (3.3) we conclude that

$$\left(\sum_{j=1}^m \overline{S}(f, (x_j, b_j), \eta_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left(\sum_{j=1}^m \overline{R}(\varphi, (x_j, b_j), \eta_j)^p \right)^{\frac{1}{p}}$$

for all $x_1, \dots, x_m \in E$, $b_1, \dots, b_m \in G$, $\eta_1, \dots, \eta_m \in \mathbb{K}$ and $m \in \mathbb{N}$.

Since \overline{R} and \overline{S} satisfy (2) and (3), from Theorem 2.1 we conclude that there is a measure μ so that

$$\overline{S}(f, (x, b), \eta) \leq C \left(\int_K \overline{R}(\varphi, (x, b), \eta)^p d\mu \right)^{1/p}$$

for all $x \in E$, $b \in G$ and $\eta \in \mathbb{K}$. Hence it easily follows that, for all $x \in E$ and $b \in G$, we have

$$S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^p d\mu \right)^{1/p}.$$

□

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